

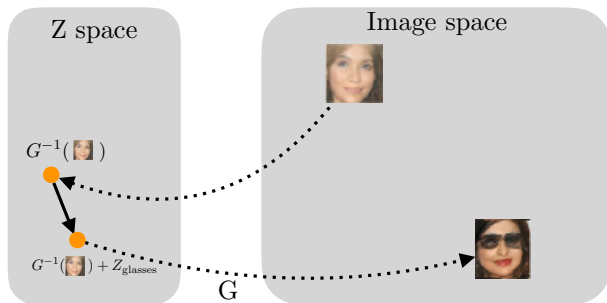
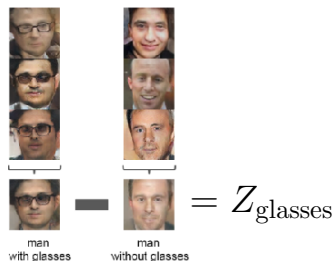
The power of two samples for Generative Adversarial Networks

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University of Illinois at Urbana-Champaign

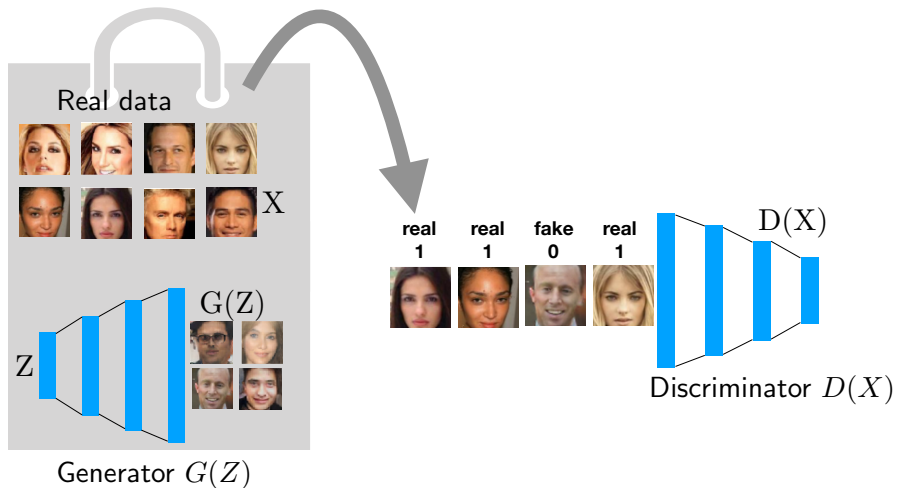
joint work with Giulia Fanti, Ashish Khetan, Zinan Lin

Generative models learn representation²



²[DCGAN, Radford et al. 2015]

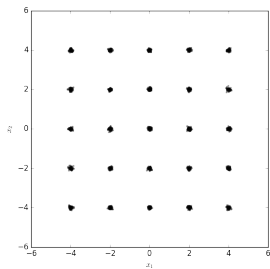
Generative Adversarial Networks (GAN)



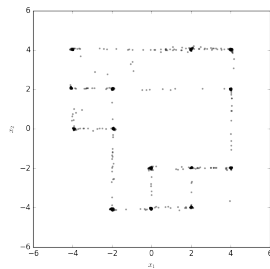
$$\min_G \max_D V(G, D)$$

Mode collapse

Target distribution



GAN

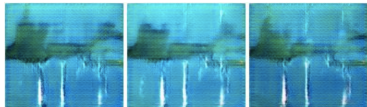


Mode collapse

- “A man in a orange jacket with sunglasses and a hat ski down a hill.”³



- “This guy is in black trunks and swimming underwater.”



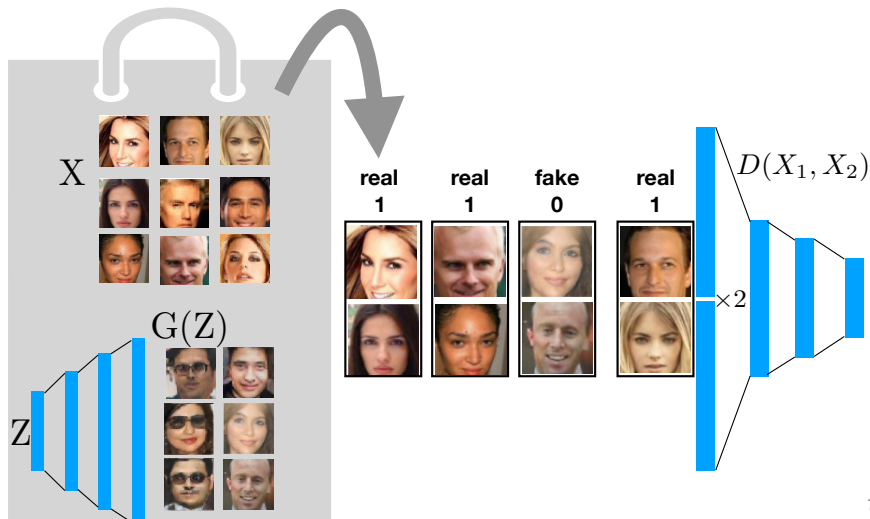
- “A tennis player in a blue polo shirt is looking down at the green court.”



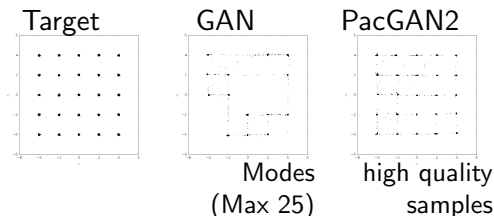
³[“Generating interpretable images with controllable structure”, by Reed et al., 2016]

New framework: PacGAN

- lightweight overhead
- experimental results
- principled

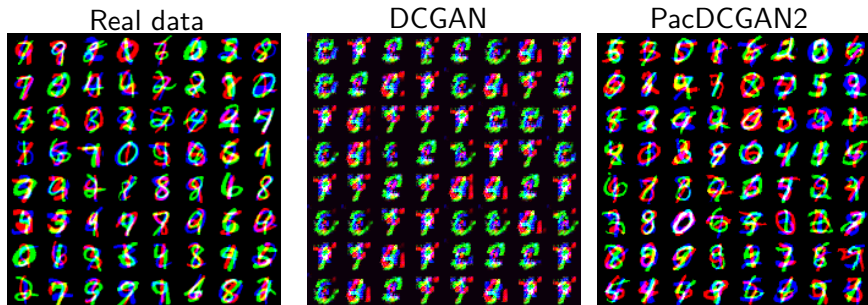


Benchmark datasets from VEEGAN paper



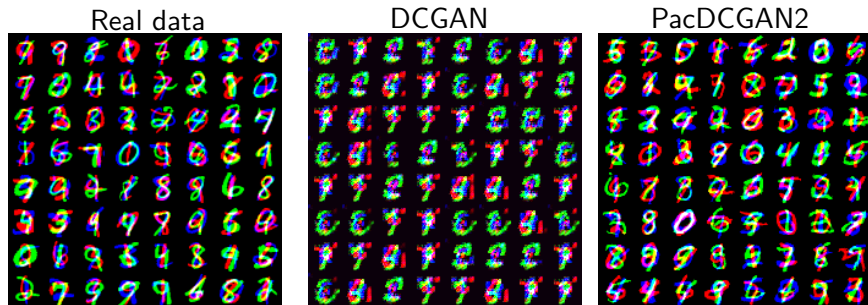
GAN	3.3	0.5 %
ALI	15.8	1.6 %
Unrolled GAN	23.6	16.0 %
VEEGAN	24.6	40.0 %
PacGAN2	24.6 ± 0.9	65.8 ± 13.4 %
PacGAN3	24.9 ± 0.3	71.4 ± 13.8 %
PacGAN4	25.0 ± 0.0	76.0 ± 7.1 %

Benchmark datasets from VEEGAN paper



	Modes (Max 1000)	KL
DCGAN	99.0	3.40
ALI	16.0	5.40
Unrolled GAN	48.7	4.32
VEEGAN	150.0	2.95
PacDCGAN2	1000.0±0.0	0.06±0.01
PacDCGAN3	1000.0±0.0	0.06±0.01
PacDCGAN4	1000.0±0.0	0.07±0.01

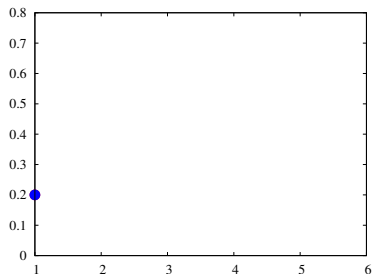
Benchmark from Unrolled GAN (small discriminator)



	Modes (Max 1000)	KL
DCGAN	30.6 ± 20.73	5.99 ± 0.42
Unrolled GAN, 1 step	65.4 ± 34.75	5.91 ± 0.14
Unrolled GAN, 5 steps	236.4 ± 63.30	4.67 ± 0.43
Unrolled GAN, 10 steps	327.2 ± 74.67	4.66 ± 0.46
PacDCGAN2	370.8 ± 244.34	3.33 ± 1.02
PacDCGAN3	534.3 ± 103.68	2.11 ± 0.52
PacDCGAN4	557.7 ± 101.37	2.06 ± 0.61

Intuition behind packing via toy example

Target distribution P

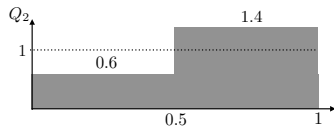


Generator Q_1
with mode collapse



$$d_{\text{TV}}(P, Q_1) = 0.2$$

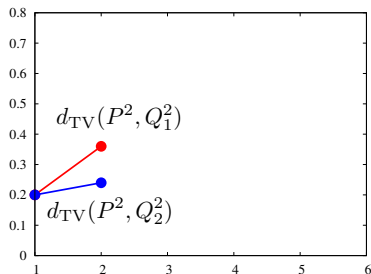
Generator Q_2
without mode collapse



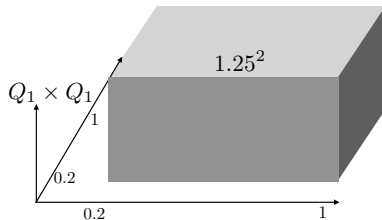
$$d_{\text{TV}}(P, Q_2) = 0.2$$

Intuition behind packing via toy example

Target distribution P

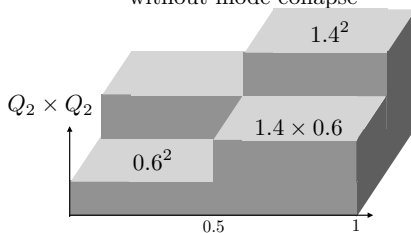


Generator Q_1
with mode collapse



$$d_{\text{TV}}(P \times P, Q_1 \times Q_1) = 0.36$$

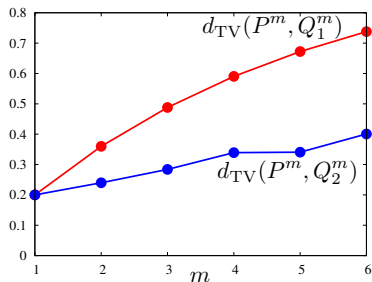
Generator Q_2
without mode collapse



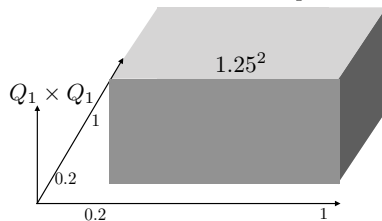
$$d_{\text{TV}}(P \times P, Q_2 \times Q_2) = 0.24$$

Intuition behind packing via toy example

Target distribution P

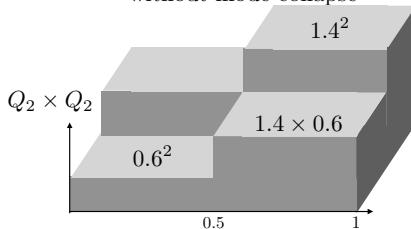


Generator Q_1
with mode collapse



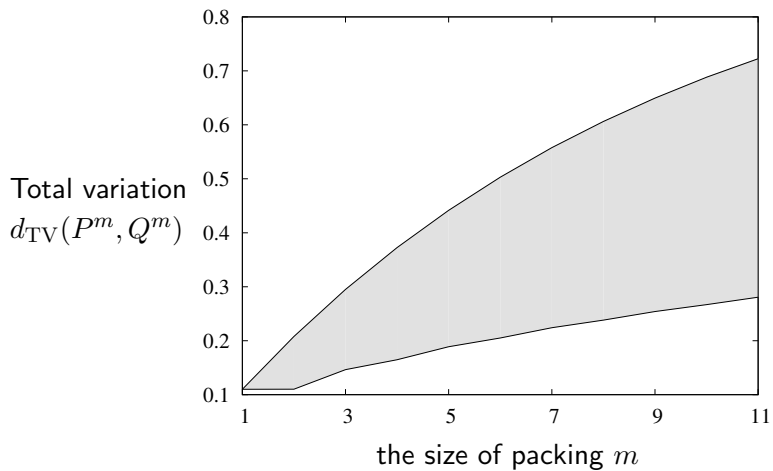
$$d_{TV}(P \times P, Q_1 \times Q_1) = 0.36$$

Generator Q_2
without mode collapse

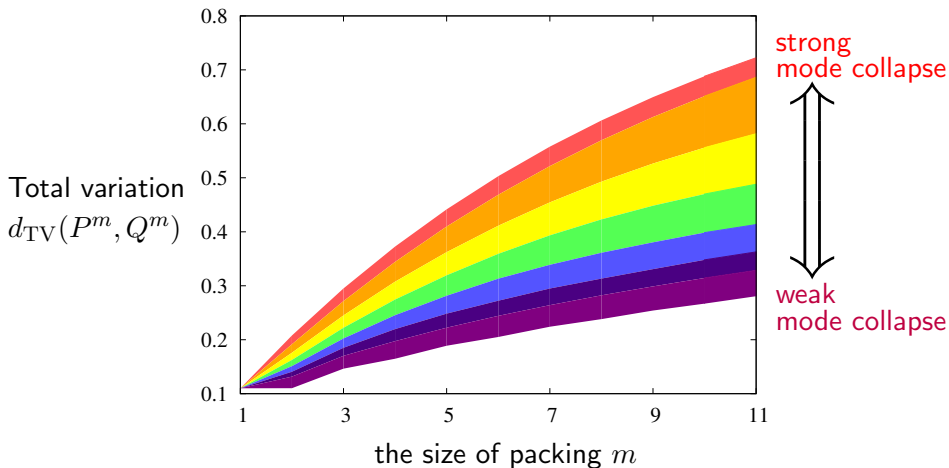


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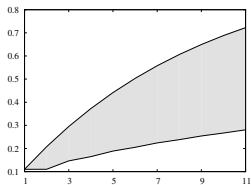
Evolution of TV distances



Evolution of TV distances through the prism of packing



Through packing, the target-generator pairs are expanded over the strengths of the mode collapse



$$\begin{array}{ll} \max_{P,Q} / \min_{P,Q} & d_{\text{TV}}(P^2, Q^2) \\ \text{subject to} & d_{\text{TV}}(P, Q) = \tau \end{array}$$

- we focus on $m = 2$ for this talk

Intuition from Blackwell

Definition [mode collapse region]

We say a pair (P, Q) of a target distribution P and a generator distribution Q has (ε, δ) -**mode collapse** if there exists a set S such that

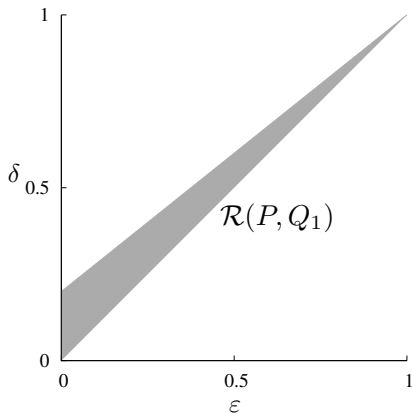
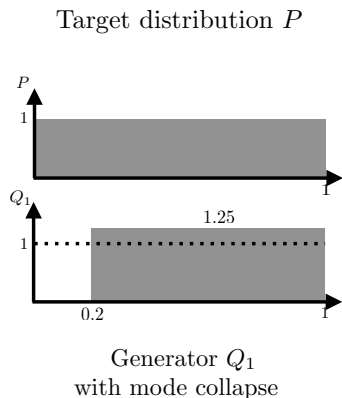
$$P(S) \geq \delta \quad , \quad \text{and} \quad Q(S) \leq \varepsilon .$$

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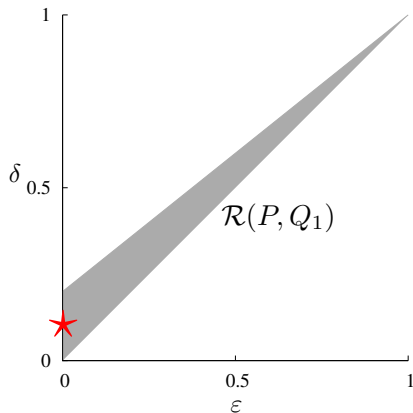
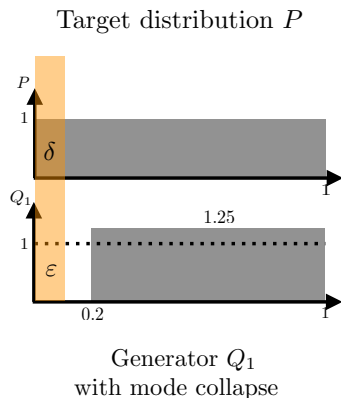


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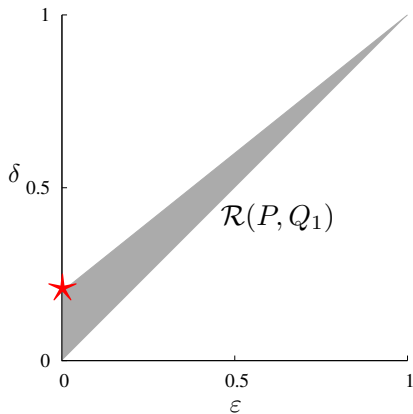
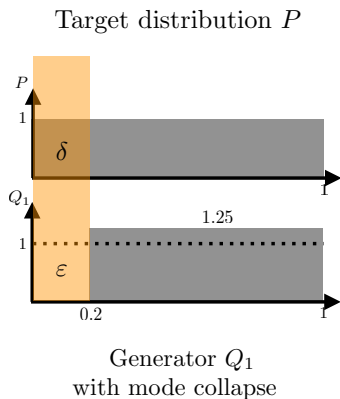


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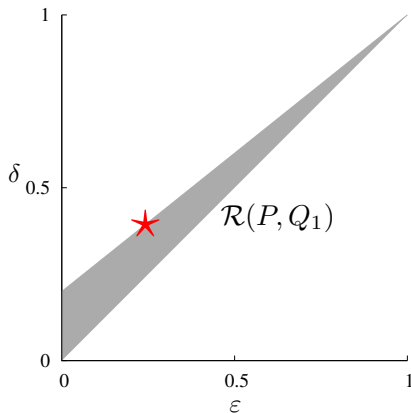
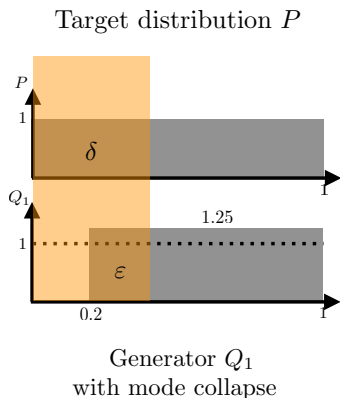


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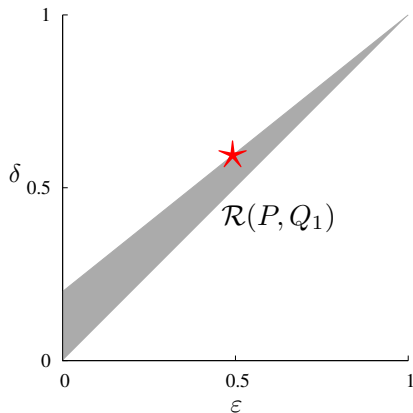
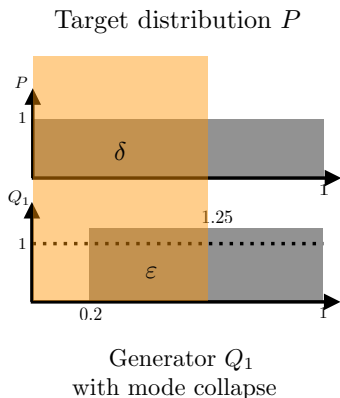


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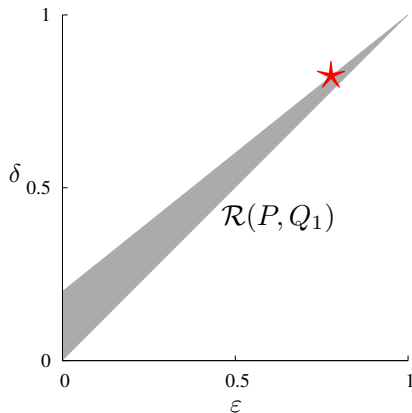
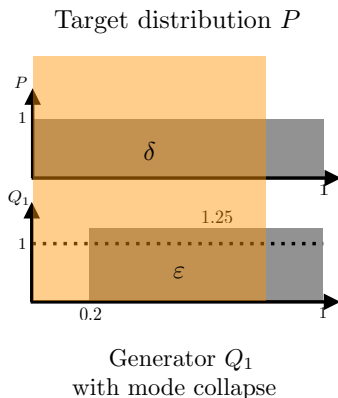


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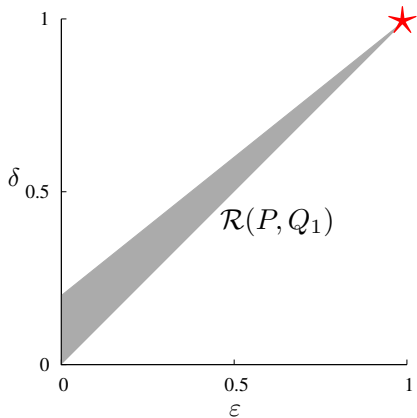
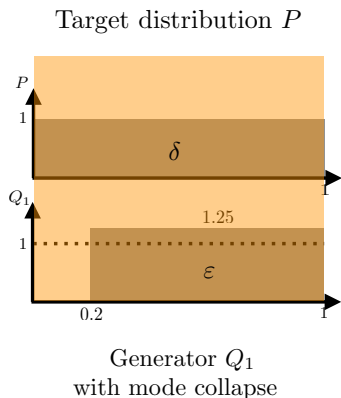


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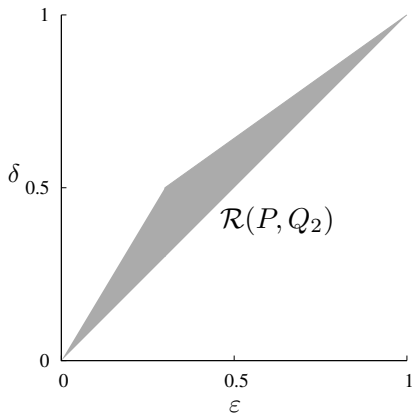
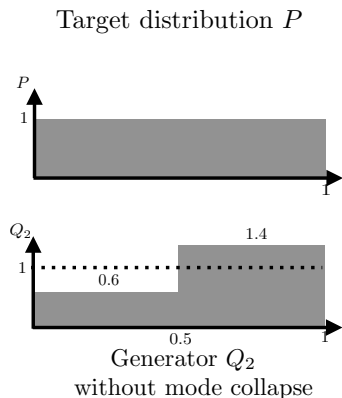


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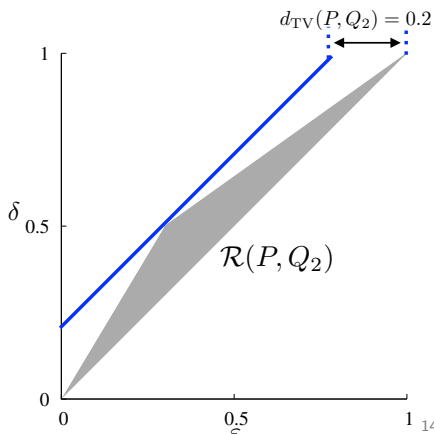
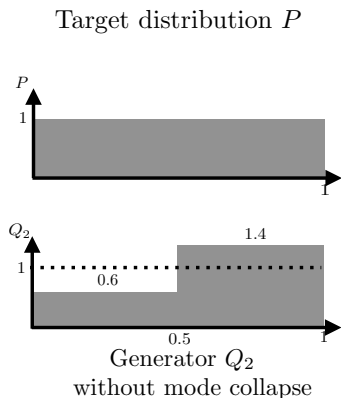


Intuition from Blackwell

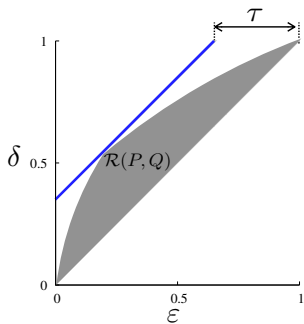
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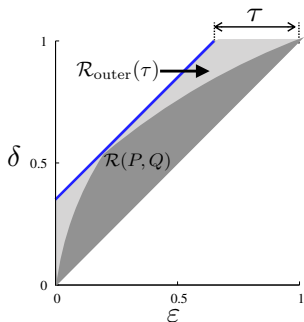


Upper bound



$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \end{aligned}$$

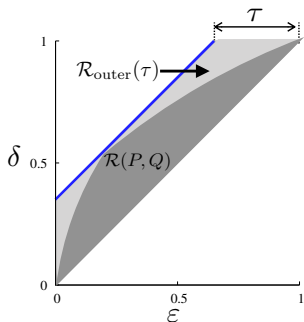
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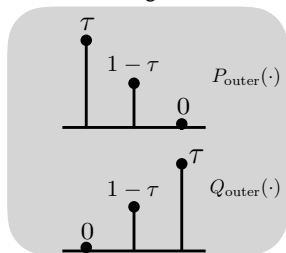
$$\mathcal{R}(P, Q) \subseteq \mathcal{R}_{\text{outer}}(\tau)$$

Upper bound

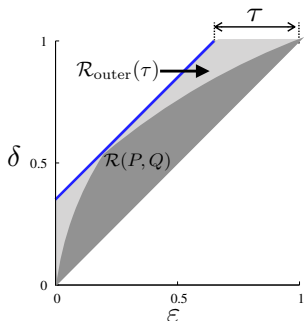


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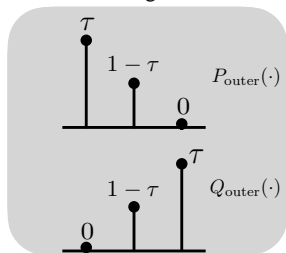


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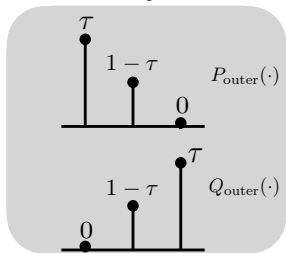
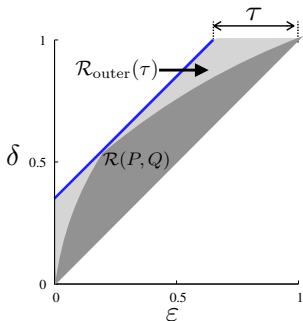
$$\begin{aligned} \mathcal{R}(P, Q) & \subseteq \mathcal{R}_{\text{outer}}(\tau) \\ \mathcal{R}(P^2, Q^2) & \subseteq \mathcal{R}(P_{\text{outer}}^2, Q_{\text{outer}}^2) \end{aligned}$$



Blackwell's theorem

$$\begin{aligned} & \mathcal{R}(P, Q) \subseteq \mathcal{R}(P', Q') \\ \Rightarrow & \mathcal{R}(P^2, Q^2) \subseteq \mathcal{R}(P'^2, Q'^2) \end{aligned}$$

Upper bound



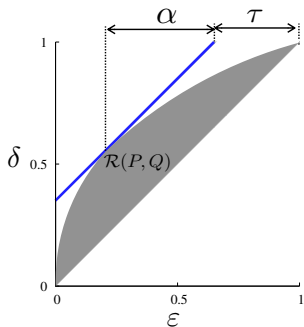
$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \end{aligned}$$

$$\begin{aligned} \mathcal{R}(P, Q) &\subseteq \mathcal{R}_{\text{outer}}(\tau) \\ \mathcal{R}(P^2, Q^2) &\subseteq \mathcal{R}(P_{\text{outer}}^2, Q_{\text{outer}}^2) \\ d_{\text{TV}}(P^2, Q^2) &\leq \underbrace{d_{\text{TV}}(P_{\text{outer}}^2, Q_{\text{outer}}^2)}_{1-(1-\tau)^2} \end{aligned}$$

Blackwell's theorem

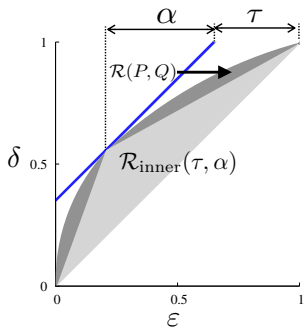
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Lower bound



$$\begin{aligned} & \min_{P, Q} d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to } d_{\text{TV}}(P, Q) = \tau \end{aligned}$$

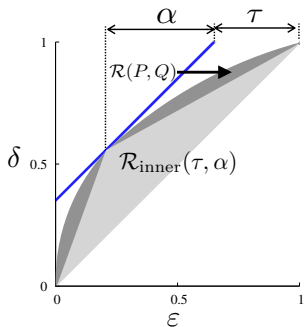
Lower bound



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$$\mathcal{R}_{\text{inner}}(\tau, \alpha) \subseteq \mathcal{R}(P, Q)$$

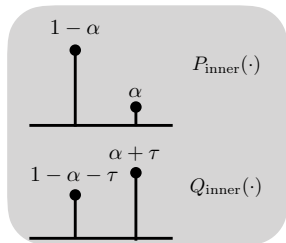
Lower bound



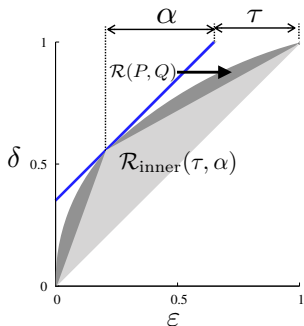
$$\min_{P, Q} d_{\text{TV}}(P^2, Q^2)$$

subject to $d_{\text{TV}}(P, Q) = \tau$

$$\mathcal{R}_{\text{inner}}(\tau, \alpha) \subseteq \mathcal{R}(P, Q)$$



Lower bound

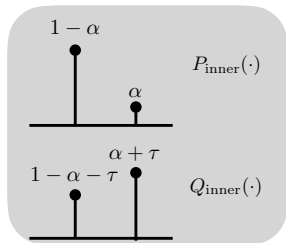


$$\min_{P, Q} d_{\text{TV}}(P^2, Q^2)$$

subject to $d_{\text{TV}}(P, Q) = \tau$

$$\mathcal{R}_{\text{inner}}(\tau, \alpha) \subseteq \mathcal{R}(P, Q)$$

$$\mathcal{R}(P_{\text{inner}}^2, Q_{\text{inner}}^2) \subseteq \mathcal{R}(P^2, Q^2)$$

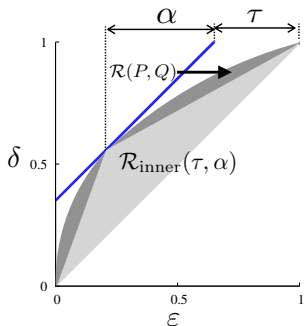


Blackwell's theorem

$$\mathcal{R}(P, Q) \subseteq \mathcal{R}(P', Q')$$

$$\Rightarrow \mathcal{R}(P^2, Q^2) \subseteq \mathcal{R}(P'^2, Q'^2)$$

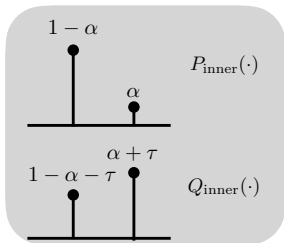
Lower bound



$$\min_{P, Q} d_{\text{TV}}(P^2, Q^2)$$

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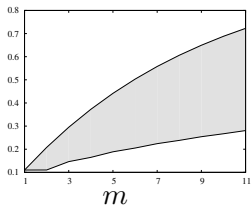
$$\begin{aligned} \mathcal{R}_{\text{inner}}(\tau, \alpha) &\subseteq \mathcal{R}(P, Q) \\ \mathcal{R}(P_{\text{inner}}^2, Q_{\text{inner}}^2) &\subseteq \mathcal{R}(P^2, Q^2) \\ \underbrace{d_{\text{TV}}(P_{\text{inner}}^2, Q_{\text{inner}}^2)}_{\tau} &\leq d_{\text{TV}}(P^2, Q^2) \end{aligned}$$



Blackwell's theorem

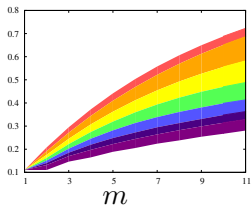
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$$d_{\text{TV}}(P^m, Q^m)$$



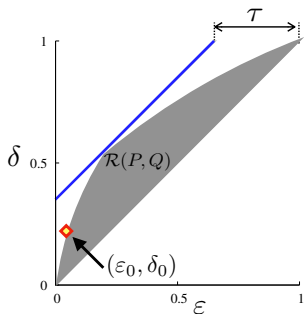
$$\begin{array}{ll} \max / \min & d_{\text{TV}}(P^2, Q^2) \\ P, Q & P, Q \\ \text{subject to} & d_{\text{TV}}(P, Q) = \tau \end{array}$$

$$d_{\text{TV}}(P^m, Q^m)$$



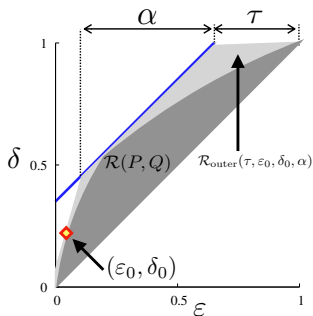
$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && \text{no } (\varepsilon_0, \delta_0)\text{-mode collapse} \end{aligned}$$

Upper bound without $(\varepsilon_0, \delta_0)$ -mode collapse



$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && \text{no } (\varepsilon_0, \delta_0)\text{-mode collapse} \end{aligned}$$

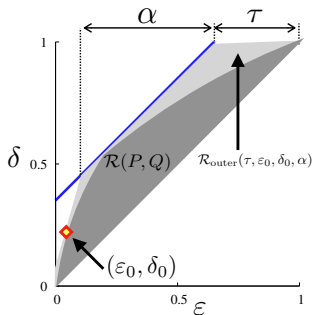
Upper bound without $(\varepsilon_0, \delta_0)$ -mode collapse



$\max_{P, Q} \quad d_{\text{TV}}(P^2, Q^2)$
subject to $d_{\text{TV}}(P, Q) = \tau$
no $(\varepsilon_0, \delta_0)$ -mode collapse

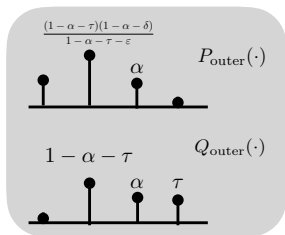
$$\mathcal{R}(P, Q) \subseteq \mathcal{R}_{\text{outer}}(\tau, \varepsilon_0, \delta_0, \alpha)$$

Upper bound without $(\varepsilon_0, \delta_0)$ -mode collapse

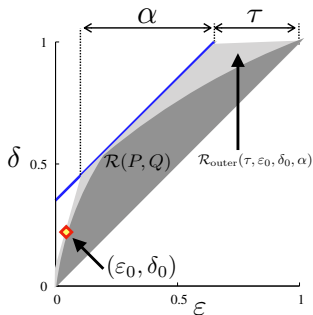


$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && \text{no } (\varepsilon_0, \delta_0)\text{-mode collapse} \end{aligned}$$

$$\mathcal{R}(P, Q) \subseteq \mathcal{R}_{\text{outer}}(\tau, \varepsilon_0, \delta_0, \alpha)$$

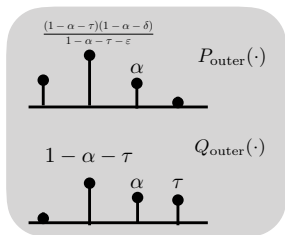


Upper bound without $(\varepsilon_0, \delta_0)$ -mode collapse



$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && \text{no } (\varepsilon_0, \delta_0)\text{-mode collapse} \end{aligned}$$

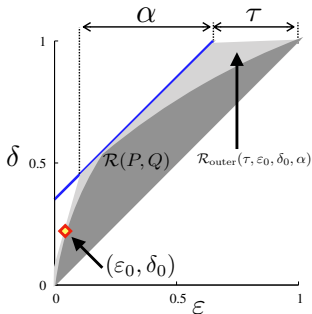
$$\begin{aligned} \mathcal{R}(P, Q) & \subseteq \mathcal{R}_{\text{outer}}(\tau, \varepsilon_0, \delta_0, \alpha) \\ \mathcal{R}(P^2, Q^2) & \subseteq \mathcal{R}(P_{\text{outer}}^2, Q_{\text{outer}}^2) \end{aligned}$$



Blackwell's theorem

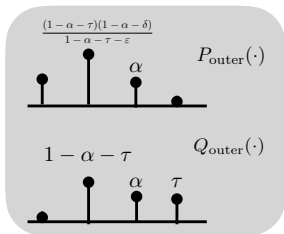
$$\begin{aligned} & \mathcal{R}(P, Q) \subseteq \mathcal{R}(P', Q') \\ \Rightarrow & \mathcal{R}(P^2, Q^2) \subseteq \mathcal{R}(P'^2, Q'^2) \end{aligned}$$

Upper bound without $(\varepsilon_0, \delta_0)$ -mode collapse



$$\begin{aligned} & \max_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && \text{no } (\varepsilon_0, \delta_0)\text{-mode collapse} \end{aligned}$$

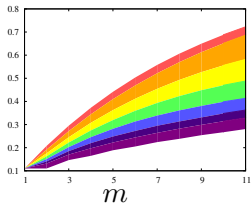
$$\begin{aligned} \mathcal{R}(P, Q) & \subseteq \mathcal{R}_{\text{outer}}(\tau, \varepsilon_0, \delta_0, \alpha) \\ \mathcal{R}(P^2, Q^2) & \subseteq \mathcal{R}(P_{\text{outer}}^2, Q_{\text{outer}}^2) \\ d_{\text{TV}}(P^2, Q^2) & \leq \underbrace{d_{\text{TV}}(P_{\text{outer}}^2, Q_{\text{outer}}^2)}_{\text{simple to evaluate}} \end{aligned}$$



Blackwell's theorem

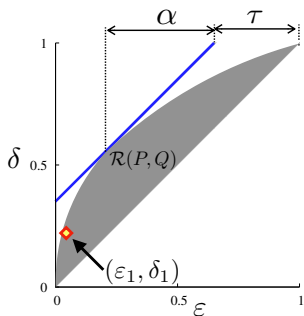
$$\begin{aligned} & \mathcal{R}(P, Q) \subseteq \mathcal{R}(P', Q') \\ \Rightarrow & \mathcal{R}(P^2, Q^2) \subseteq \mathcal{R}(P'^2, Q'^2) \end{aligned}$$

$$d_{\text{TV}}(P^2, Q^2)$$



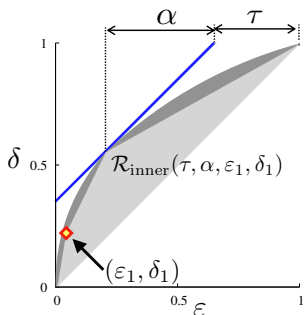
$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

Lower bound with $(\varepsilon_1, \delta_1)$ -mode collapse



$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

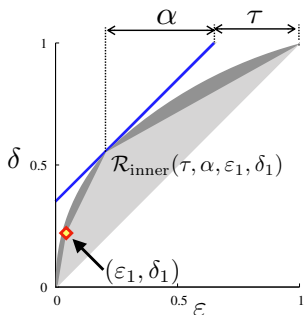
Lower bound with $(\varepsilon_1, \delta_1)$ -mode collapse



$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

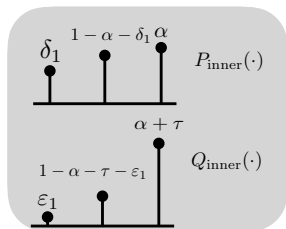
$$\mathcal{R}_{\text{inner}}(\tau, \alpha, \varepsilon_1, \delta_1) \subseteq \mathcal{R}(P, Q)$$

Lower bound with $(\varepsilon_1, \delta_1)$ -mode collapse

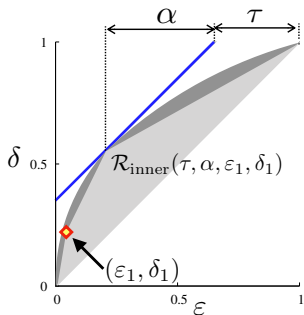


$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

$$\mathcal{R}_{\text{inner}}(\tau, \alpha, \varepsilon_1, \delta_1) \subseteq \mathcal{R}(P, Q)$$



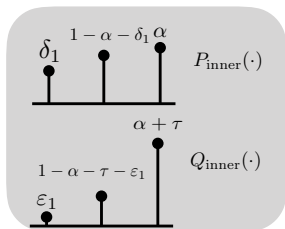
Lower bound with $(\varepsilon_1, \delta_1)$ -mode collapse



$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

$$\mathcal{R}_{\text{inner}}(\tau, \alpha, \varepsilon_1, \delta_1) \subseteq \mathcal{R}(P, Q)$$

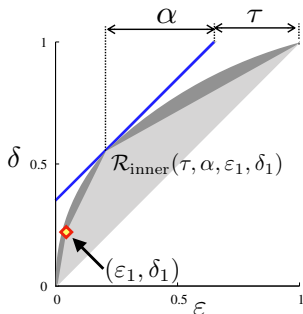
$$\mathcal{R}(P_{\text{inner}}^2, Q_{\text{inner}}^2) \subseteq \mathcal{R}(P^2, Q^2)$$



Blackwell's theorem

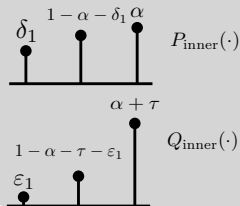
$$\begin{aligned} & \mathcal{R}(P, Q) \subseteq \mathcal{R}(P', Q') \\ \Rightarrow & \mathcal{R}(P^2, Q^2) \subseteq \mathcal{R}(P'^2, Q'^2) \end{aligned}$$

Lower bound with $(\varepsilon_1, \delta_1)$ -mode collapse



$$\begin{aligned} & \min_{P, Q} && d_{\text{TV}}(P^2, Q^2) \\ & \text{subject to} && d_{\text{TV}}(P, Q) = \tau \\ & && (\varepsilon_1, \delta_1)\text{-mode collapse} \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{\text{inner}}(\tau, \alpha, \varepsilon_1, \delta_1) & \subseteq \mathcal{R}(P, Q) \\ \mathcal{R}(P_{\text{inner}}^2, Q_{\text{inner}}^2) & \subseteq \mathcal{R}(P^2, Q^2) \\ \underbrace{d_{\text{TV}}(P_{\text{inner}}^2, Q_{\text{inner}}^2)}_{\text{simple to evaluate}} & \leq d_{\text{TV}}(P^2, Q^2) \end{aligned}$$



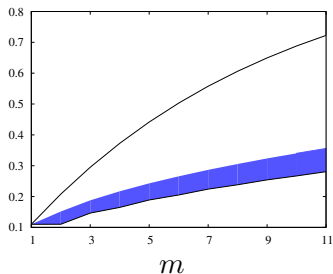
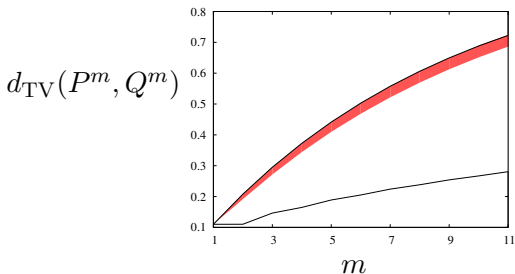
Blackwell's theorem

$$\begin{aligned} \mathcal{R}(P, Q) & \subseteq \mathcal{R}(P', Q') \\ \Rightarrow \mathcal{R}(P^2, Q^2) & \subseteq \mathcal{R}(P'^2, Q'^2) \end{aligned}$$

Achievable TV distances for distributions

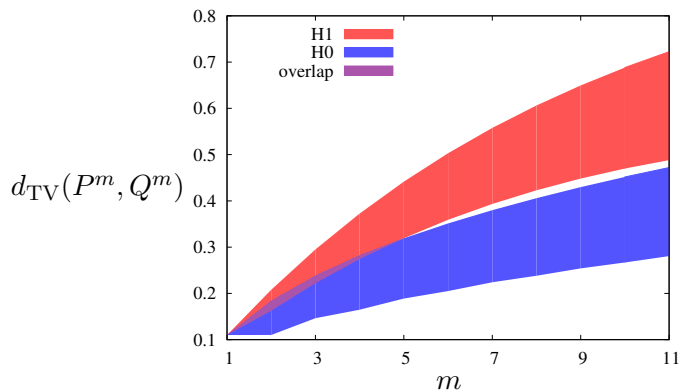
with $(\varepsilon_1, \delta_1)$ -mode collapse

without $(\varepsilon_0, \delta_0)$ -mode collapse

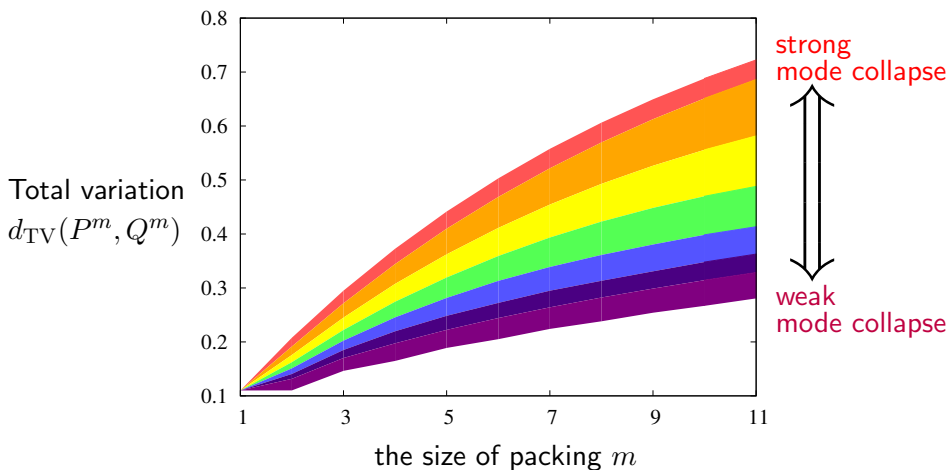


with packing, the discriminator naturally penalizes (P, Q) with severe mode collapses

Does larger m help?



Evolution of TV distances through the prism of packing



Through the **prism** packing, the target-generator pairs are expanded over the **spectrum** of (the strengths of) mode collapse

Our paper is available at:
<https://arxiv.org/abs/1712.04086>

- *Packing* is a principled solution to mode collapse
 - ▶ Lightweight overhead
 - ▶ Excellent numerical results
 - ▶ Theoretical analysis
- Exciting time for bridging the theory and practice of GAN
 - “Generalization and Equilibrium in Generative Adversarial Nets”, Sanjeev Arora, Rong Ge, Yingyu Liang, Tengyu Ma, Yi Zhang
 - “Approximation and Convergence Properties of Generative Adversarial Learning”, Shuang Liu, Olivier Bousquet, Kamalika Chaudhuri
 - “Compressed Sensing using Generative Models”, Ashish Bora, Ajil Jalal, Eric Price, Alexandros G. Dimakis
 - “Towards Understanding the Dynamics of Generative Adversarial Networks”, Jerry Li, Aleksander Madry, John Peebles, Ludwig Schmidt



Giulia Fanti (CMU)



Ashish Khetan (UIUC)



Zinan Lin (CMU)